Using EM for Reinforcement Learning

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Abstract

We discsus Hinton's (1989) *relative payoff procedure* (RPP), a static reinforcement learning algorithm whose foundation is not stochastic gradient ascent. We show circumstances under which applying the RPP is guaranteed to increase the mean return, even though it can make large changes in the values of the parameters. The proof is based on a mapping between the RPP and a form of the expectation-maximisation procedure of Dempster, Laird & Rubin (1976).

1 Introduction

Consider a stochastic learning automaton (*eg* Narendra & Thatachar, 1989) whose actions y are drawn from a set $\mathcal{Y}$. This could, for instance, be the set of 2^n choices over n separate binary decisions. If the automaton maintains a probability distribution $p(y|\theta)$ over these possible actions, where θ is a set of parameters, then its task is to learn values of the parameters θ that maximize the expected payoff:

$$\rho(\theta) = \langle \mathcal{E}[r|y] \rangle_{p(y|\theta)} = \sum_{\mathcal{Y}} p(y|\theta)\mathcal{E}[r|y]. \tag{1}$$

Here, $\mathcal{E}[r|y]$ is the expected reward for performing action y.

Apart from random search, simulated annealing and genetic algorithms, almost all the methods with which we are familiar for attempting to choose appropriate θ in such domains are local in the sense that they make small steps usually in a direction that bears some relation to the gradient. Examples include the Keifer-Wolfowitz procedure (Wasan, 1969) the A_{RP} algorithm (Barto & Anandan, 1985) and the REINFORCE framework (Williams, 1992). There are two reasons to make small steps, one is that there is a noisy *estimation* problem – typically the automaton will emit single actions $y^1, y^2, \ldots$ according to $p(y|\theta)$, will receive single samples of the reward from the distributions $p(r|y^m)$, and will have to average the gradient over these noisy values. The other is that θ might have a complicated effect on which actions are chosen, or the relationship between actions and rewards might be obscure. For instance, if $\mathcal{Y}$ is the set of 2^n choices over n separate binary actions $a_1 \ldots a_n$, and $\theta = \{p_1, \ldots, p_n\}$ is the collection of probabilities of choosing $a_i = 1$, so

$$p(y = \{a_1..a_n\}|\theta) = \prod_{i=1}^{n} p_i^{a_i}(1-p_i)^{1-a_i}, \tag{2}$$

then the average reward $\rho(\theta)$ depends in a complicated manner on the collection of p_i. Gradient ascent in θ would seem the only option as taking large steps might lead to decreases in the average reward.

The sampling problem is indeed present, and we will evade it by assuming a large batch size. However, we show that in circumstances such as the n binary action task, it is possible to make large well-founded changes to the parameters without explicitly estimating the curvature of the space of expected payoffs, by a mapping onto a maximum likelihood probability density estimation problem. In effect, we maximize reward by solving a sequence of probability matching problems, where θ is chosen at each step to match as best it can a fictitious distribution that is determined by the average rewards experienced on the previous step. Although there can be large changes in θ from one step to the next, we are guaranteed that the average reward is monotonically increasing. The guarantee comes for exactly the same reason as in the expectation-maximization (EM) algorithm (Baum, Petrie, Soules & Weiss, 1970; Dempster, Laird & Rubin, 1977) and, as with EM, there can be local optima. The relative payoff procedure (RPP) (Hinton, 1989) is a particular reinforcement learning algorithm for the n binary action task with positive r which makes large moves in the p_i. Our proof demonstrates that the RPP is well-founded.

2 Theory

The RPP operates to improve the parameters $p_1 \dots p_n$ of equation 2 in a synchronous manner based on substantial sampling. It suggests updating the probability of choosing $a_i = 1$ to:

$$p'_i = \frac{\langle a_i \mathcal{E}[r|y] \rangle_{p(y|\theta)}}{\langle \mathcal{E}[r|y] \rangle_{p(y|\theta)}} \tag{3}$$

which is the ratio of the mean reward that accrues when $a_i = 1$ to the net mean reward. If all the rewards r are positive, then $0 \le p'_i \le 1$. This note proves that using the RPP, the expected reinforcement increases, *ie* $\langle \mathcal{E}[r|y] \rangle_{p(y|\theta')} \ge \langle \mathcal{E}[r|y] \rangle_{p(y|\theta)}$ where $\theta' = \{p'_1, \dots, p'_n\}$.

The proof rests on the following observation. Given a current value of θ, if one could arrange that:

$$\alpha \mathcal{E}[r|y] = \frac{p(y|\theta')}{p(y|\theta)} \tag{4}$$

for some α, then θ' would lead to higher average returns than θ. We prove this formally below, but the intuition is that if $\mathcal{E}[r|y_1] > \mathcal{E}[r|y_2]$, then $\frac{p(y_1|\theta')}{p(y_2|\theta')} > \frac{p(y_1|\theta)}{p(y_2|\theta)}$, so θ' will put more weight on y_1 than θ does. We therefore pick θ' so that $p(y|\theta')$ *matches* the distribution $\alpha \mathcal{E}[r|y] p(y|\theta)$ as well as possible (using a Kullback-Leibler penalty). Note that this target distribution moves with θ. Matching just $\beta \mathcal{E}[r|y]$, something that animals can be observed to do under some circumstances (Gallistel, 1990) does *not* result in maximizing average rewards (Sabes & Jordan, 1995).

If the rewards r are stochastic, then our method (and the RPP) does not eliminate the need for repeated sampling to work out the mean return. We assume knowledge of $\mathcal{E}[r|y]$. Defining the distribution in equation 4 correctly requires $\mathcal{E}[r|y] > 0$. Since maximizing $\rho(\theta)$ and $\rho(\theta) + \omega$ have the same consequences for θ, we can add arbitrary constants to the rewards so that they are all positive. However, this can affect the rate of convergence.

We now show how an improvement in the expected reinforcement can be guaranteed:

$$\begin{aligned}
\log \frac{\rho(\theta')}{\rho(\theta)} &= \log \sum_{y \in \mathcal{Y}} p(y|\theta') \frac{\mathcal{E}[r|y]}{\rho(\theta)} \\
&= \log \sum_{y \in \mathcal{Y}} \left[\frac{p(y|\theta)\mathcal{E}[r|y]}{\rho(\theta)} \right] \frac{p(y|\theta')}{p(y|\theta)} \qquad (5) \\
&\geq \sum_{y \in \mathcal{Y}} \left[\frac{p(y|\theta)\mathcal{E}[r|y]}{\rho(\theta)} \right] \log \frac{p(y|\theta')}{p(y|\theta)}, \text{ by Jensen's inequality} \\
&= \frac{1}{\rho(\theta)} \left[Q(\theta, \theta') - Q(\theta, \theta) \right].
\end{aligned} \qquad (6)$$

where

$$Q(\theta, \theta') = \sum_{y \in \mathcal{Y}} p(y|\theta)\mathcal{E}[r|y] \log p(y|\theta')$$

so if $Q(\theta, \theta') \geq Q(\theta, \theta)$, then $\rho(\theta') \geq \rho(\theta)$. The normalization step in equation 5 creates the matching distribution from equation 4. Given θ, if θ' is chosen to maximize $Q(\theta, \theta')$, then we are guaranteed that $Q(\theta, \theta') \geq Q(\theta, \theta)$ and therefore that the average reward is non-decreasing.

In the RPP, the new probability p'_i for choosing $a_i = 1$ is given by:

$$p'_i = \frac{\langle a_i \mathcal{E}[r|y] \rangle_{p(y|\theta)}}{\langle \mathcal{E}[r|y] \rangle_{p(y|\theta)}}, \tag{7}$$

so it is the fraction of the average reward that arrives when $a_i = 1$. Using equation 2,

$$\frac{\partial Q(\theta, \theta')}{\partial p'_i} = \frac{1}{p'_i(1 - p'_i)} \left[\sum_{y \in \mathcal{Y}: a_i = 1} p(y|\theta)\mathcal{E}[r|y] - p'_i \sum_{y \in \mathcal{Y}} p(y|\theta)\mathcal{E}[r|y] \right]$$

So, if:

$$p'_i = \frac{\sum_{y \in \mathcal{Y}: a_i = 1} p(y|\theta)\mathcal{E}[r|y]}{\sum_{y \in \mathcal{Y}} p(y|\theta)\mathcal{E}[r|y]}, \text{ then } \frac{\partial Q(\theta, \theta')}{\partial p'_i} = 0,$$

and it is readily seen that $Q(\theta, \theta')$ is maximized. But this condition is just that of equation 7. Therefore the RPP is monotonic in the average return.

Figure 1 shows the consequence of employing the RPP. The left diagram (a) shows the case in which $n = 2$; the two lines and associated points show how p_1 and p_2 change on successive steps using the RPP. The terminal value $p_1 = p_2 = 0$ reached by the left-hand line is a local optimum. Note that the RPP always changes the parameters in the direction of the gradient of the expected amount of reinforcement (this is generally true), but by a variable amount.

Figure 1b compares the RPP with (a deterministic version of) Williams' (1992) stochastic gradient ascent REINFORCE algorithm for a case with $n = 12$ and rewards $\mathcal{E}[r|y]$ drawn from an exponential distribution. The RPP and REINFORCE were started at the same point; the graph shows the difference between the maximum possible reward and the expected reward after given numbers of iterations. To make a fair comparison between the two algorithms we chose n small enough that the exact averages in equation 3 and (for REINFORCE) the exact gradients:

$$p'_i = \alpha \frac{\partial}{\partial p_i} \langle \mathcal{E}[r|y] \rangle_{p(y|\theta)}$$

a) b)

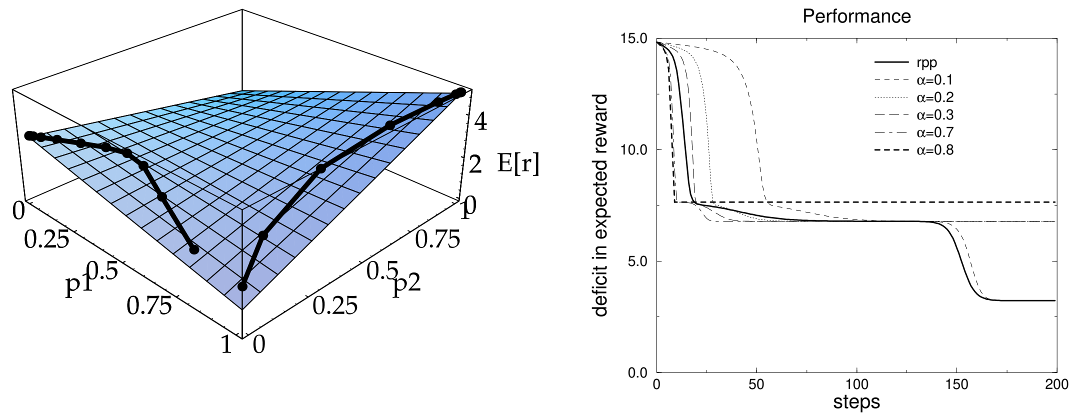

Figure 1: Performance of the RPP. a) Adaptation of p_1 and p_2 using the RPP from two different start points on the given $\rho(p_1, p_2)$. The points are succesive values; the lines are joined for graphical convenience. b) Comparison of the RPP with William's (1992) REINFORCE for a particular problem with $n = 12$. See text for comment and details.

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could be calculated. Figure 1b shows the (consequently smooth) course of learning for various values of the learning rate. We observe that for both algorithms, the expected return never decreases (as guaranteed for the RPP but not REINFORCE), that the course of learning is not completely smooth – with a large plateau in the middle, and that both algorithms get stuck in local minima. This is a best case for the RPP – only for a small learning rate and consequently slow learning does REINFORCE not get stuck in a worse local minimum. On other cases, there are values of α for which REINFORCE beats the RPP. However, there are no free parameters in the RPP, and it performs well across a variety of such problems.

3 Discussion

The analogy to EM can be made quite precise. EM is a maximum likelihood method for probability density estimation for a collection $\mathcal{X}$ of observed data where underlying point $x \in \mathcal{X}$ there can be a *hidden variable* $y \in \mathcal{Y}$. The density has the form:

$$p(x|\theta) = \sum_{y \in \mathcal{Y}} p(y|\theta) p(x|y, \theta)$$

and we seek to choose θ to maximize

$$\sum_{x \in \mathcal{X}} \log \left[p(x|\theta) \right]$$

The E-phase of EM calculates the posterior responsibilities $p(y|x, \theta)$ for each $y \in \mathcal{Y}$ for each x:

$$p(y|x, \theta) = \frac{p(y|\theta) p(x|y, \theta)}{\sum_{z \in \mathcal{Y}} p(z|\theta) p(x|z, \theta)}$$

In our case, there is no x, but the equivalent of this posterior distribution, which comes from equation 4, is

$$\mathcal{P}_y(\theta) \equiv \frac{p(y|\theta) \mathcal{E}[r|y]}{\sum_{z \in \mathcal{Y}} p(z|\theta) r(z)}.$$

The M-phase of EM chooses parameters θ' in the light of this posterior distribution to maximize

$$\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(y|x, \theta) \log[p(x, y|\theta')].$$

In our case this is exactly equivalent to minimizing the Kullback-Leibler divergence

$$KL[\mathcal{P}_y(\theta), p(y|\theta')] = -\sum_{y \in \mathcal{Y}} \mathcal{P}_y(\theta) \log \left[\frac{p(y|\theta')}{\mathcal{P}_y(\theta)} \right]$$

Up to some terms that do not affect θ', this is $-Q(\theta, \theta')$. The Kullback-Leibler divergence between two distributions is a measure of the distance between them, and therefore minimizing it is a form of probability matching.

Our result is weak – the requirement for sampling from the distribution is rather restrictive, and we have not proved anything about the actual rate of convergence. The algorithm performs best (Sutton, personal communication) if the differences between the rewards are of the same order of magnitude as the rewards themselves (as a result of the normalization in equation 5) – it uses multiplicative comparison rather than the subtractive comparison of Sutton (1984; Williams, 1992; Dayan, 1990).

The link to the EM algorithm suggests that there may be reinforcement learning algorithms other than the RPP which make large changes to the values of the parameters for which similar guarantees about non-decreasing average rewards can be given. The most interesting extension would be to dynamic programming (Bellman, 1957) whose techniques for choosing (single-component) actions to optimize return in sequential decision tasks include two algorithms that make large changes on each step: value and policy iteration (Howard, 1960). As various people have noted, the latter explicitly involves both *estimation* (of the value of a policy) and *maximization* (choosing a new policy in the light of the value of each state under the old one), although its theory is not at all described in terms of density modeling.

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